

THE PRODUCT OF TWO NORMAL INITIALLY κ -COMPACT SPACES

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ABSTRACT. We prove that it is independent from ZFC that for every cardinal κ the following statements are equivalent:

- (a) κ is singular;
- (b) initial κ -compactness (defined above the introduction) is productive;
- (c) initial κ -compactness is finitely productive; and
- (d) the product of two initially κ -compact normal spaces is initially κ -compact.

In particular, MA implies that there are two countably compact normal spaces whose product is not countably compact.

In our terminology T_1 implies regular and normal, and κ , λ , and μ denote infinite cardinals. Recall that a space is called *initially κ -compact* if every open cover of cardinality at most κ has a finite subcover, and that initial ω -compactness is equivalent to *countable compactness*, for T_1 -spaces. The meaning of *initially $< \kappa$ -compact* should be clear.

1. INTRODUCTION

The Tychonoff Product Theorem asserts that the product of any number of compact spaces is compact. Since a space X is compact iff it is initially $|X|$ -compact this suggests that one consider analogous statements for products of initially κ -compact spaces, for various κ .

Novák and Teresaka showed, independently, that there exist two initially ω -compact regular (even zero-dimensional) spaces whose product is not initially ω -compact [N, T]. These spaces are not normal, at least under CH (see §3). This suggests the natural question, perhaps due to Bandy (cf. [R, Problem B15]), of whether there exists an example in which the two spaces are normal. Our first result is that MA (= Martin's Axiom) implies that this is the case.

1.1. Theorem. *Assume MA. There exist two initially $< \mathfrak{c}$ -compact normal spaces E_0 and E_1 such that $E_0 \times E_1$ is not even initially ω -compact. (In fact, even $E_0 \times E_1$ is normal; see §8.)*

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1.2. Corollary. *Assume MA. There exist two countably compact normal spaces whose product is not countably compact.* \square

This is the original motivation for this paper. I do not know if MA is essential in 1.2. However, MA is essential for 1.1, as will be shown below. My original proof of this fact used a certain compactness-like property; I am indebted to Jerry Vaughan for identifying this property as initial ω_1 -compactness; this was the starting point for what follows.

We now turn our attention to the productivity of initial κ -compactness for $\kappa > \omega$. The following positive result is due to Stephenson and Vaughan [SV, Theorem 1.1]; it was proved earlier by Saks and Stephenson with the additional condition that all factors are regular [SS, Theorem 4.11].

1.3. Theorem. *The product of any number of initially κ -compact spaces is initially κ -compact provided κ is singular and $2^\lambda < \kappa$ for all $\lambda < \kappa$.* \square

Saks and Stephenson ask whether [SS, p. 281], or not [SS, p. 287], initial κ -compactness is productive for (regular) $\kappa > \omega$. Under GCH we answer this question as follows.

1.4. Theorem. *Assume GCH. For every regular κ there exist two initially κ -compact normal spaces whose product is not initially κ -compact.*

An obvious corollary to Theorems 1.3 and 1.4 is

1.5. Theorem. *Assume GCH. The following are equivalent for each cardinal κ :*

- (a) κ is singular;
- (b) the product of any number of initially κ -compact spaces is initially κ -compact;
- (c) the product of two initially κ -compact spaces is initially κ -compact; and
- (d) the product of two normal initially κ -compact spaces is initially κ -compact. \square

I do not know if Theorem 1.4 can be proved in ZFC. However, the following corollary to Theorem 1.1 shows that Theorem 1.5 is not true in ZFC.

1.6. Theorem. *Assume $MA + \mathfrak{c} > \omega_\omega$. There are two initially ω_ω -compact normal spaces whose product is not initially ω_ω -compact.* \square

Note that in light of Theorem 1.5 this also shows that Theorem 1.1 is not true in ZFC (see also Remark 2.7) and that in fact we have shown that the statement that “initial κ -compactness is productive for all singular κ ” is independent from ZFC. However, this result would be just another way of saying that GCH is consistent and independent if the condition in Theorem 1.3 is necessary, i.e., if the fact that initial κ -compactness is productive implies both that κ is singular and that $2^\lambda < \kappa$ for all $\lambda < \kappa$. This remains an open question.

A trivial but unexpected and interesting corollary to Theorem 1.5 is our last theorem:

1.7. Theorem. *Assume GCH. Then for every cardinal κ the following implication holds:*

F_κ : *if initial κ -compactness is finitely productive, then it is productive.* \square

I do not know if this is true in ZFC. To appreciate the difficulty of this question one should note that F_ω is (vacuously) true, and recall that Frolik has given an example of a space Y such that ${}^\omega Y$ is not initially ω -compact, yet ${}^n Y$ is initially ω -compact for all $n < \omega$ [Fr, Theorem B].

I am also indebted to Jerry Vaughan for spotting a number of inaccuracies in the manuscript of this paper.

2. PRELIMINARIES

An ordinal is the set of smaller ordinals, and a cardinal is an initial ordinal. If A is a set, the set of functions $A \rightarrow 2$ is denoted by ${}^A 2$ and carries the product topology (2 is discrete). For a set A and a cardinal ν we put as usual

$$[A]^\nu = \{X \subseteq A: |X| = \nu\};$$

$[A]^{\leq \nu}$ and $[A]^{< \nu}$ are defined analogously. Also, for a set A we define

$$H(A) = \bigcup \{ {}^F 2: F \in [A]^{< \omega} \};$$

note that $\{\{y \in {}^A 2: x \subseteq y\}: x \in H(A)\}$ is a base for ${}^A 2$.

If A is a set, an A -sequence is a function f with domain A , and is denoted by $\langle f_a: a \in A \rangle$.

A space X is called κ -bounded if every member of $[X]^{\leq \kappa}$ has compact closure [GFW]; we define $< \kappa$ -bounded analogously. We remind the reader of the following facts.

2.1. Proposition. (a) *The product of any number of κ -bounded spaces is κ -bounded [GFW, Lemma 4].*

(b) *Every κ -bounded space is initially κ -compact [GFW, Lemma 3].*

(c) *Every regular initially 2^κ -compact space is κ -bounded [SS, Theorem 2.1].*

(d) *If κ is singular, then every initially $< \kappa$ -compact space is initially κ -compact, implicit in [SS, proof of Theorem 4.11]. \square*

Each cardinal carries the discrete topology. As usual we identify the points of $\beta\kappa$ with free ultrafilters on κ ; in particular, if $\alpha \in \kappa$ then we do not distinguish α from $\{A \subseteq \kappa: \alpha \in A\}$.

We are particularly interested in the open subspace of $\beta\kappa$ consisting of the nonuniform ultrafilters, i.e., in

$$\text{NU}(\kappa) = \{p \in \beta\kappa: \exists P \in p (|P| < \kappa)\}.$$

Note that $\text{NU}(\kappa) = \kappa$ if (and only if) $\kappa = \omega$. We define the set of (small) unbounded subsets of $\text{NU}(\kappa)$ by

$$\text{UB}(\kappa) = \{I \in [\text{NU}]^{\text{cf}(\kappa)}: \text{the closure of } I \text{ in } \text{NU}(\kappa) \text{ is not compact}\}.$$

2.2. Fact. $\text{NU}(\kappa)$ is $< \text{cf}(\kappa)$ -bounded but not $\text{cf}(\kappa)$ -bounded. \square

2.3. Fact. For every $A \subseteq \text{NU}(\kappa)$ whose closure in $\text{NU}(\kappa)$ is not compact there is $I \in \text{UB}(\kappa)$ with $I \subseteq A$. \square

We omit the easy proofs. We will use Fact 2.3 in the construction of the examples, together with the following easy

2.4. Fact. Assume GCH. Then $|\text{UB}(\kappa)| = \kappa^+$, for all $\kappa > \omega$.

Proof. $|[\kappa]^{<\kappa}| \leq \kappa^+$ and $|\beta\lambda| = 2^{2^\lambda} \leq \kappa^+$ if $\lambda < \kappa$, hence $\kappa \leq |\text{NU}(\kappa)| \leq \kappa^+$, under GCH, so $|\text{UB}(\kappa)| \leq \kappa^+$.

To see that $|\text{UB}(\kappa)| \geq \kappa^+$, let $A = \{p_\alpha: \alpha < \kappa\} \subset \text{NU}(\kappa)$ be such that for each $\omega \leq \alpha < \kappa$, $p_\alpha \in \beta\alpha$ and $\forall P \in p_\alpha (|P| = |\alpha|)$. Any unbounded subset of A of size $\text{cf}(\kappa)$ is in $\text{UB}(\kappa)$, and there are $\kappa^{\text{cf}(\kappa)} = \kappa^+$ many of them. \square

2.5. Remark. Actually one can easily show that $|\text{NU}(\omega)| = \omega$, and, if $\kappa > \omega$, then $|\text{NU}(\kappa)| = \kappa^\kappa \cdot \sum_{\lambda < \kappa} 2^{2^\lambda}$, in ZFC.

The proof of our final piece of preliminary information is analogous to the proof of the special case $b\kappa = \beta\kappa$, hence is omitted.

2.6. Fact. Let $b\kappa$ be any compactification of κ with $\text{NU}(\kappa) \subseteq b\kappa$. If K is a clopen (= closed and open) subset of $b\kappa - \text{NU}(\kappa)$, then there is a $A \subseteq \kappa$, unique up to a set of cardinality $< \kappa$, such that $\overline{A} - \text{NU}(\kappa) = K$ and \overline{A} is clopen in $b\kappa$.

(Here “unique up to a set of cardinality $< \kappa$ ” means that $\overline{B} - \text{NU}(\kappa) = K$ iff $|(A - B) \cup (B - A)| < \kappa$, for all $B \subseteq \kappa$).

2.7. Remark. In the introduction we gave one proof that Theorem 1.1 requires MA. Here we give another: It is consistent with ZFC that \mathfrak{c} be singular [Ku]. But if \mathfrak{c} is singular then the product of any number of initially $< \mathfrak{c}$ -compact regular spaces is countably compact, as one can easily prove from Proposition 2.1.

These two proofs require $\mathfrak{c} > \omega_\omega$. One can also show that Theorem 1.1 requires MA even if $\mathfrak{c} = \omega_2$. Indeed, it is consistent with $\text{ZFC} + \mathfrak{c} = \omega_2$ that the product of any number of initially ω_1 -compact spaces be countably compact [vD, Theorem 7.3].

3. THE BASIC IDEA. I

In this section we explain the basic idea of how to get two countably compact normal spaces whose product is not countably compact.

Novák constructs two countably compact spaces X_i with $\omega \subset X_i \subset \beta\omega$ (\subset denotes proper inclusion) for $i < 2$ such that

(0) $X_0 \cap X_1 = \omega$.

Then $X_0 \times X_1$ is not countably compact since $\{\langle n, n \rangle: n < \omega\}$ is a countable subset of $X_0 \times X_1$ which is closed in $X_0 \times X_1$, being the intersection of $X_0 \times X_1$ and the diagonal $\{\langle x, x \rangle: x \in \beta\omega\}$ of the Hausdorff space $\beta\omega$. Under CH the X_i 's are not normal, for otherwise $X_i - \omega$ would be C^* -embedded in X_i , hence in $\beta X_i = \beta\omega$, hence in $\beta\omega - \omega$; also $X_i - \omega$ is a proper subset of $\beta\omega - \omega$ which is dense in $\beta\omega - \omega$ since X_i is countably compact. But under CH there are no dense proper C^* -embedded subspaces in $\beta\omega - \omega$ [FG, Theorem 4.6]. So $\beta\omega$ is the wrong compactification of ω to use.

We plan to construct a different compactification $b\omega$ of ω and two suitable subspaces Σ_0 and Σ_1 of $b\omega - \omega$ such that

(1) $\Sigma_0 \cap \Sigma_1 = \emptyset$.

Then our examples will be the subspaces

$$E_i = \omega \cup \Sigma_i \quad (i < 2)$$

of $b\omega$. Then we get the fact that $E_0 \times E_1$ is not countably compact for free, for $E_0 \cap E_1 = \omega$, hence $E_0 \times E_1$ is not countably compact, as above.

We now investigate the question of how one can ensure that E_0 and E_1 are countably compact and normal. We first consider countable compactness. The easy proofs of the following fact and its corollary are omitted.

3.1. Fact. Both E_i 's are countably compact if and only if

(2) both Σ_i 's are countably compact; and

(3) $\forall i < 2 \forall I \in [\omega]^\omega (\bar{I} \cap \Sigma_i) \neq \emptyset$. \square

3.2. Corollary. If both E_i 's are countably compact, then

(4) both Σ_i 's are dense in $b\omega - \omega$. \square

We next consider normality. Since disjoint closed sets of any space have disjoint neighborhoods iff their boundaries have disjoint neighborhoods, a moment's reflection shows that the following holds.

3.3. Fact. A sufficient condition for the E_i 's to be normal is

(5) every two disjoint closed sets in Σ_i have disjoint closures in $b\omega - \omega$ ($i < 2$). \square

Note that this only involves the way the Σ_i 's sit inside $b\omega - \omega$, and not the way the Σ_i 's are attached to ω (cf. (3)).

3.4. Remark. Note that (5) implies that the Σ_i 's are normal, and if (4) holds, then (5) holds iff the Σ_i 's are normal and $\beta\Sigma_i = b\omega - \omega$ ($i < 2$).

So if we can ensure that (2), (3), and (5) hold, then our E_i 's are countably compact and normal. We do not want to have to consider three conditions simultaneously. Therefore the first step of the construction is to pick a compact space P having subspaces Σ_0 and Σ_1 such that (2) and the analogues of (4) and (5), with P instead of $b\omega - \omega$, hold. The second step is to show that ω has a compactification $b\omega$ with $b\omega - \omega = P$ such that (3) holds. (Note that we need (4) since it is a necessary condition for (3) to hold.) It is just a happy coincidence that E_0 and E_1 are not only countably compact but even initially $< \mathfrak{c}$ -compact.

My first result was that this plan works (but that one needs MA in the second step). This has been the key to the discovery of Theorem 1.4, whose basic idea will be explained in the next section. One does not have to know yet how we perform our two steps. However, it may be helpful to know that there is a purely set-theoretic translation of the existence of $b\omega$, so that all one has to do in the second step is to construct a certain family of subsets of ω .

4. THE BASIC IDEA. II

Let κ be a regular cardinal. Assuming enough additional axioms we will construct a compactification $b\kappa$ of κ and initially κ -compact subspaces E_0 and E_1 such that $E_0 \times E_1$ is not initially κ -compact.

As before, we ensure that $E_0 \times E_1$ is not initially κ -compact by making $E_0 \cap E_1$ not initially compact. (This is why E_0 and E_1 must be subspaces of the same Hausdorff space.) We cannot have $E_0 \cap E_1 = \kappa$, unless $\kappa = \omega$. For we will have to assume that $2^\lambda \leq \kappa$ for $\lambda < \kappa$, hence E_0 and E_1 are $< \kappa$ -bounded by 2.1(c), hence so is $E_0 \times E_1$, by 2.1(a).

We choose $E_0 \times E_1$ before we begin the construction. Since we are interested in a $b\kappa$ whose existence is equivalent to the existence of some family of subsets of κ (like in the special case $\kappa = \omega$), we choose $E_0 \cap E_1 = \text{NU}(\kappa)$. Note that this agrees with our choice for $\kappa = \omega$.

So $b\kappa$ is a compactification of $\text{NU}(\kappa)$ as well. We must have suitable subspaces Σ_0 and Σ_1 of $b\kappa - \text{NU}(\kappa)$, then our examples will be the subspaces

$$E_i = \text{NU}(\kappa) \cup \Sigma_i \quad (i < 2)$$

of $b\kappa$. (It will not be necessary to give the E_i 's and Σ_i 's the subscript κ .)

We now investigate when our examples are initially λ -compact; the reason for considering the cases $\lambda \neq \kappa$ is that in Theorem 1.1 we consider initial $< \mathfrak{c}$ -compactness, and because of Remark 7.2.

4.1. Fact. Let $\lambda \geq \kappa$, then the following are equivalent:

- (a) Both E_0 and E_1 are initially λ -compact.
- (b) The following two statements hold:
 - (1) both Σ_0 and Σ_1 are initially λ -compact; and
 - (2) $\forall i < 2 \forall I \in \text{UB}(\kappa) (\bar{I} \cap \Sigma_i \neq \emptyset)$.

Proof. That (a) implies (b) is easy to see. (The assumption $\lambda \geq \text{cf}(\kappa)$ is used to get (2) from (a).) To prove that (b) implies (a) let $i < 2$ and let \mathcal{U} be an open cover of E_i with $|\mathcal{U}| \leq \lambda$. There is a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $\Sigma_i \subseteq \bigcup \mathcal{V}$. There is no $I \in \text{UB}(\kappa)$ with $I \subseteq \text{NU}(\kappa) - \bigcup \mathcal{V}$, hence by Fact 2.3 the subspace $\text{NU}(\kappa) - \bigcup \mathcal{V}$ is compact. Hence \mathcal{U} has a finite subcover. \square

The following fact is similar to Fact 3.3, but the proof is slightly less trivial since $E_i - \Sigma_i$ is no longer discrete if $\kappa > \omega$.

4.2. Fact. A sufficient condition for the E_i 's to be normal is that (2) and the following statement hold:

- (3) every two disjoint closed sets in Σ_i have disjoint closures in $b\kappa - \text{NU}(\kappa)$ ($i < 2$).

Proof. Fix $i < 2$. We prove that E_i is normal by proving

- (α) any two disjoint closed subsets of E_i have disjoint closures in $b\kappa$.

To see this observe that (2) implies that for any $A \subset \text{NU}(\kappa)$, $\bar{A} - \text{NU}(\kappa) = \overline{(\bar{A} \cap \Sigma_i)}$, where the overbar is the closure operator in $b\kappa$. (Use Fact 2.3 here.) So if F is a closed subset of E_i ,

$$\bar{F} = \overline{(F \cap \Sigma_i)} \cup (F \cap \text{NU}(\kappa)). \quad \square$$

4.3. Remark. We just showed that (2) and (3) imply (α), or, equivalently, that the E_i 's are normal and that $\beta E_i = b\kappa$ ($i < 2$). It is easy to see that conversely (α) implies (2) and (3).

5. OUR CHOICE FOR $b\kappa - \text{NU}(\kappa)$ AND THE Σ_i 's

In the previous two sections we have seen that, given a regular κ , we need a compact space P , having subspaces Σ_0 and Σ_1 , such that

- (1) $\Sigma_0 \cap \Sigma_1 = \emptyset$,
- (2) Σ_i is initially κ -compact ($i < 2$),
- (3) Σ_i is dense in P ($i < 2$),
- (4) disjoint closed subsets of Σ_i have disjoint closures in P ($i < 2$).

(Without condition (1) this is trivial.) In all real examples that I am aware of P is a product of compact spaces and the Σ_i 's are generalized Σ -products. We will consider the case where P is a product of 2-point spaces: Given a (regular) κ , we plan to have

$$b\kappa - \text{NU}(\kappa) = {}^{(2^\kappa)}2$$

and

$$\Sigma_i = \{x \in {}^{(2^\kappa)}2 : |\{\alpha < 2^\kappa : x(\alpha) \neq i\}| < 2^\kappa\}.$$

It is obvious that (1) and (3) hold. It follows from the discussion below that (2) and (4) hold, and that in fact

(2') Σ_i is initially $< \text{cf}(2^\kappa)$ -compact.

The discussion is somewhat more general than is really needed since this is easier.

5.1. Convention. For the remainder of this section λ and μ are cardinals with $\omega < \lambda \leq \mu$, and Σ is the subspace

$$\Sigma = \{x \in {}^\mu 2 : |\{\alpha < \mu : x(\alpha) \neq 0\}| < \lambda\}$$

of ${}^\mu 2$, and for $A \subseteq \mu$ we identify ${}^A 2$ in the natural way with

$$\{x \in {}^\mu 2 : \{\alpha < \mu : x(\alpha) \neq 0\} \subseteq A\}.$$

5.2. Proposition. (a) Σ is $< \text{cf}(\lambda)$ -bounded (hence is initially $< \text{cf}(\lambda)$ -compact).

(b) Σ is not initially $\text{cf}(\lambda)$ -compact.

Proof. If $X \subseteq \Sigma$ has cardinality $< \text{cf}(\lambda)$, then

$$A = \{\alpha < \mu : x(\alpha) \neq 0 \text{ for some } x \in X\}$$

has cardinality less than λ , hence ${}^A 2 \subseteq \Sigma$. Since $X \subseteq {}^A 2$, it follows that X has compact closure. This proves (a).

Next let C be a cofinal subset of λ with $|C| = \text{cf}(\lambda)$. Then

$$\{\{x \in \Sigma : x(\alpha) = 0 \text{ for some } \alpha < \gamma\} : \gamma \in C\}$$

is an open cover of Σ which has no finite subcover. This proves (b). \square

5.3. Theorem. If λ is regular, then every two disjoint closed subsets of Σ have disjoint closures in ${}^\mu 2$.

Proof. This follows from the proofs of [C, Theorem 1; Ko, Theorem 1; KoM, Basic Lemma]. We sketch the proof for the convenience of the reader.

For $x \in {}^\mu 2$ let $\text{supt}(x) = \{\alpha \in \mu : x(\alpha) = 1\}$. Assume that $H, K \subseteq \Sigma$ and that $p \in {}^\mu 2$ is such that $p \in \overline{H} \cap \overline{K}$. By recursion, construct in ω steps a set $A \subseteq \mu$ with $|A| < \lambda$ so that for every neighborhood U of $p \upharpoonright A$ in ${}^A 2$ there are points $h \in H$ and $k \in K$ with $\text{supt}(h) \cup \text{supt}(k) \subseteq A$ and $\{h \upharpoonright A, k \upharpoonright A\} \subseteq U$.

Then $q \in {}^\mu 2$, defined by $q \upharpoonright A = p \upharpoonright A$ and $\forall \alpha \in \mu - A (q(\alpha) = 0)$, is in Σ and $q \in \overline{H} \cap \overline{K}$. \square

5.4. Remark. Instead of Σ one can consider

$$\{x \in {}^\mu 2 : |\{\alpha < \mu : x(\alpha) \neq p(\alpha)\}| < \lambda\}$$

for any $p \in {}^\mu 2$.

5.5. *Remark.* Σ is not normal if $\text{cf} \lambda = \omega$. For then Σ is not countably compact by 5.2(b), but is pseudocompact since it has the dense subspace

$$\{x \in {}^\mu 2: |\{\alpha < \mu: x(\alpha) \neq 0\}| < \omega_1\},$$

which is countably compact by 5.2(a). I did not investigate whether or not Σ is normal if $\omega < \text{cf}(\lambda) < \lambda$.

We need GCH (or MA if $\kappa = \omega$) not only to construct our spaces, but also to prove that E_0 and E_1 are normal since each 2^κ is (trivially) regular under GCH, and 2^ω is regular under MA. The latter fact also implies that E_0 and E_1 are initially $< \mathfrak{c}$ -compact if $\kappa = \omega$.

6. A SET-THEORETIC TRANSLATION

We are interested in finding a compactification $b\kappa$ of κ such that

- (1) $\text{NU}(\kappa) \subseteq b\kappa$ and $b\kappa - \text{NU}(\kappa) = ({}^{2^\kappa}2)$; and
- (2) $\forall i < 2 \forall I \in \text{UB}(\kappa) (\bar{I} \cap \Sigma_i \neq \emptyset)$.

It will be convenient to have a simple set-theoretic translation of the existence of such a compactification. The following notation is useful to cut down on the length of the formulas. Recall that $H(\alpha) = \bigcup \{ {}^F 2: F \in [\alpha]^{<\omega} \}$.

6.1. *Notation.* If $\langle X_\xi: \xi < \alpha \rangle$ is an α -sequence of subsets of a set S , then for every $f \in H(\alpha)$ we define $X(S, f) \subseteq S$ by

$$X(S, f) = \{s \in S: \forall \xi \in \text{dom}(f) (s \in X_\xi \Leftrightarrow f(\xi) = 1)\},$$

or, equivalently,

$$X(S, f) = \bigcap \{X_\xi: \xi \in \text{dom}(f) \wedge f(\xi) = 1\} \cap \bigcap \{S - X_\xi: \xi \in \text{dom}(f) \wedge f(\xi) = 0\}$$

if $f \neq \emptyset$, and $X(S, \emptyset) = S$.

6.2. **Lemma.** *The following statements are equivalent for a regular cardinal κ :*

- (a) κ has a compactification $b\kappa$ for which (1) and (2) holds; and
- (b) there is a 2^κ -sequence $\langle X_\xi: \xi < 2^\kappa \rangle$ of subsets of κ such that
 - (3) $\forall f \in H(2^\kappa) [|X(\kappa, f)| = \kappa]$; and
 - (4) $\forall i < 2 \forall I \in \text{UB}(\kappa) \exists x \in \Sigma_i \forall f \in H(2^\kappa) [f \subseteq x \Rightarrow \forall \eta < \kappa \exists y \in I (X(\kappa, f) - \eta \in y)]$.

Proof. We only need the implication (b) \rightarrow (a), but the proof of the reverse implication is felt to help in understanding what is going on.

We first show how a compactification $b\kappa$ of κ satisfying (1) gives rise to a 2^κ -sequence $\langle X_\xi: \xi < 2^\kappa \rangle$ of subsets of κ satisfying (3) and conversely. Once we are able to switch back and forth it is easy to see that (2) and (4) are mere translations of each other.

Remember that $\text{NU}(\kappa)$ is $< \kappa$ -bounded since κ is regular, by 2.2.

For $\xi < 2^\kappa$ define the subbasic clopen set $K_\xi \subseteq ({}^{2^\kappa}2)$ by

$$K_\xi = \{x \in ({}^{2^\kappa}2): x(\xi) = 1\}.$$

And for $f \in H({}^{(2^\kappa)}2)$, let $K({}^{(2^\kappa)}2, f) = \{x \in ({}^{2^\kappa}2): f \subseteq x\}$.

Assume $b\kappa$ is a compactification of κ satisfying (1). By Fact 2.6 there is for each $\xi < 2^\kappa$ an $X_\xi \subseteq \kappa$, unique up to a set of cardinality less than κ , such that

(5) $\overline{X}_\xi \cap {}^{(2^\kappa)}2 = K_\xi$, and \overline{X}_ξ is clopen.

For every $f \in H(2^\kappa)$ we have $X(\kappa, f)^- \cap {}^{(2^\kappa)}2 = K({}^{(2^\kappa)}2, f) \neq \emptyset$, hence $|X(\kappa, f)| = \kappa$ since $\text{NU}(\kappa)$ is $< \kappa$ -bounded. This proves (3). Observe that

(6) for every $x \in {}^{(2^\kappa)}2$ the family

$$\{(X(\kappa, f) - A)^- : f \in H(2^\kappa) \wedge f \subseteq x \wedge A \in [\kappa]^{<\kappa}\}$$

is a neighborhood base for x in $b\kappa$.

Next assume a 2^κ -sequence $\langle X_\xi : \xi < 2^\kappa \rangle$ of subsets of κ is given that satisfies (3). We let $b\kappa$ be the smallest compactification of κ for which every member of

$$\mathcal{A} = \{X_\xi : \xi < 2^\kappa\} \cup [\kappa]^{<\kappa}$$

has open closure in $b\kappa$. The best way to think of $b\kappa$ is as follows: let \mathcal{B} be the subalgebra of $\mathcal{P}(\kappa)$, the field of all subsets of κ , generated by \mathcal{A} , and let $b\kappa$ be the Stone space of \mathcal{B} . Recall that points of $b\kappa$ are ultrafilters in \mathcal{B} , so if we identify $\alpha \in \kappa$ with $\{B \in \mathcal{B} : \alpha \in B\}$ (as we did when we identified $b\kappa$ with the space of all ultrafilters on κ), then $b\kappa$ becomes a compactification of κ . With a similar identification one gets $\text{NU}(\kappa) \subseteq b\kappa$. It remains to show that there is a natural homeomorphism $h: {}^{(2^\kappa)}2 \rightarrow (b\kappa - \text{NU}(\kappa))$. One can define a function $h: {}^{(2^\kappa)}2 \rightarrow b\kappa$ by

$$h(x) = \{X(\kappa, f) - A : f \in H(2^\kappa) \wedge f \subseteq x \wedge A \in [\kappa]^{<\kappa}\}.$$

It is easy to see, using (3), that this is well defined, i.e., that each $h(x)$ is an ultrafilter of \mathcal{B} , and that h is a homeomorphism of ${}^{(2^\kappa)}2$ onto $b\kappa - \text{NU}(\kappa)$. So we will from now on identify $b\kappa - \text{NU}(\kappa)$ with ${}^{(2^\kappa)}2$. Note that our definition of h tells us that (5) and (6) hold for the $b\kappa$ we constructed from the X_ξ 's.

In summary, given $b\kappa$ we choose the X_ξ 's such that (5) holds. In turn the X_ξ 's uniquely determine $b\kappa$; this is where we used the fact that each X_ξ is unique up to a set of cardinality less than κ . This completes the description of the interrelation of $b\kappa$ and the X_ξ 's. It remains to show that (2) and (4) are equivalent.

Let $i < 2$, $I \in \text{UB}(\kappa)$, and $x \in \Sigma_i$ be given. It follows from (5) that

$$x \in \overline{I} \text{ iff } \forall f \in H(2^\kappa) \forall A \in [\kappa]^{<\kappa} (f \subseteq x \Rightarrow ((X(\kappa, f) - A)^- \cap I \neq \emptyset)),$$

hence, since κ is regular,

$$x \in \overline{I} \text{ iff } \forall f \in H(2^\kappa) (f \subseteq x \Rightarrow (\forall \eta < \kappa)((X(\kappa, f) - \eta)^- \cap I \neq \emptyset)).$$

But if $y \in \text{NU}(\kappa)$ and $T \subseteq \kappa$, then $y \in \overline{T}$ iff $T \in y$ since $\text{NU}(\kappa)$ consists of ultrafilters on κ . This proves that (2) and (4) are equivalent, as required. \square

7. THE CONSTRUCTION

Let κ be a regular cardinal, $\kappa \geq \omega$. We want to construct a certain 2^κ -sequence $\langle X_\xi : \xi < 2^\kappa \rangle$ of subsets of κ . We change Notation 6.1 by writing $X(f)$ instead of $X(\kappa, f)$. Recall that we want our X_ξ 's to satisfy

- (1) $\forall f \in H(2^\kappa) [|X(f)| = \kappa]$; and
- (2) $\forall i < 2 \forall I \in \text{UB}(\kappa) \exists x \in \Sigma_i \forall f \in H(2^\kappa) [f \subseteq x \Rightarrow \forall \eta < \kappa \exists y \in I (X(f) - \eta \in y)]$.

We assume that GCH holds, or that MA holds if $\kappa = \omega$. In both cases $|\text{UB}(\kappa)| = 2^\kappa$ by Fact 2.4. Hence we can list $\text{UB}(\kappa)$ as $\langle I_\xi: \xi < 2^\kappa \rangle$.

With transfinite recursion on $\xi < 2^\kappa$ we will construct $x_{i,\xi} \in \Sigma_i$ ($i < 2$) and $X_\xi \subseteq \kappa$. We will be interested in the following conditions:

(A $_\alpha$) $\forall f \in H(\alpha)$ [$|X(f)| = \kappa$]; and

(B $_\alpha$) $\forall i < 2 \forall \xi < \alpha \forall f \in H(\alpha)$ [$f \subseteq x_{i,\xi} \Rightarrow \forall \eta < \kappa \exists y \in I_\xi (X(f) - \eta \in y)$].

Let $\beta \leq 2^\kappa$, and assume $x_{i,\xi}$ ($i < 2$) and X_ξ to be constructed for all $\xi < \beta$ in such a way that (A $_{\alpha+1}$) and (B $_{\alpha+1}$) hold for all $\alpha < \beta$. Then (A $_\beta$) and (B $_\beta$) hold. So if $\beta = 2^\kappa$, then our construction is complete, and (1) and (2) hold. Now assume $\beta < 2^\kappa$.

Claim. There is $u \in {}^\beta 2$ such that

(3) $\forall f \in H(\beta)$ [$f \subseteq u \Rightarrow \forall \eta < \kappa \exists y \in I_\beta (X(f) - \eta \in y)$].

Proof. Let $\mathcal{U} \in \beta\kappa - \text{NU}(\kappa)$ with $\mathcal{U} \in \bar{I}_\beta$. Define $u: \beta \rightarrow 2$ by $u(\xi) = 1$ iff $X_\xi \in \mathcal{U}$. \square

We now define $x_{i,\beta}$ ($i < 2$) by

$$x_{i,\beta} = u \cup (2^\kappa - \beta) \times \{i\}.$$

Then the following truncated version of (B $_{\beta+1}$) holds:

(B') $\forall i < 2 \forall \xi \leq \beta \forall f \in H(\beta)$ [$f \subseteq x_{i,\xi} \Rightarrow \forall \eta < \kappa \exists y \in I_\xi (X(f) - \eta \in y)$].

We must construct X_β in such a way that the $(\beta+1)$ -sequence $\langle X_\xi: \xi < \beta+1 \rangle$ satisfies (A $_{\beta+1}$) and (B $_{\beta+1}$). For both conditions we only have to consider functions $f \in H(\beta+1)$ with $\beta \in \text{dom}(f)$, for if $\beta \notin \text{dom}(f)$ then we know that the statement between square brackets is satisfied because of (A $_\beta$) and (B'). Since $|\beta| < 2^\kappa$, we see that we have to construct X_β in such a way that less than 2^κ conditions on X_β (or $\kappa - X_\beta$) are satisfied.

We slightly rephrase these conditions so as to emphasize the information we have from (A $_\beta$) and (B'):

Conditions of type (A $_{\beta+1}$). Given $f \in H(\beta)$ we know that $X(f) = \kappa$, and we want $|X(f) \cap X_\beta| = |X(f) - X_\beta| = \kappa$, or, equivalently,

$$\forall \eta < \kappa \exists \gamma, \delta \in X(f) [\gamma, \delta > \eta \wedge \gamma \in X_\beta \wedge \delta \notin X_\beta].$$

Conditions of type (B $_{\beta+1}$). Given $i < 2$, $\xi \leq \beta$, $f \in H(\beta)$ with $f \subseteq x_{i,\xi}$ we know that for each $\eta < \kappa$ there is $y_\eta \in I_\xi$ with $X(f) - \eta \in y_\eta$. Since y_η is a nonuniform ultrafilter on κ it follows that there is $Y_\eta \in y_\eta \cap [\kappa]^{<\kappa}$ with $Y_\eta \subseteq X(f) - \eta$.

So it would be sufficient for our purposes to know that

$$\forall \eta < \kappa \exists \eta', \eta'' < \kappa [\eta', \eta'' \geq \eta \wedge (Y_{\eta'} \subseteq \kappa - X_\beta) \wedge (Y_{\eta''} \subseteq X_\beta)].$$

Since then $Y_{\eta'} \subseteq X(f \cup \{\langle \beta, 0 \rangle\})$, hence $X(f \cup \{\langle \beta, 0 \rangle\}) - \eta \in Y_{\eta'} \in I_\xi$; and $Y_{\eta''} \subseteq X(f \cup \{\langle \beta, 1 \rangle\})$, so $X(f \cup \{\langle \beta, 1 \rangle\}) - \eta \in Y_{\eta''} \in I_\xi$.

If we assume GCH, then there are κ such conditions to meet; since κ is regular we can find X_β with an easy construction, adding in κ steps singleton sets or elements of $[\kappa]^{<\kappa}$ to X_β or $\kappa - X_\beta$ or both.

If we assume $\kappa = \omega$ and MA, then our conditions tell us that X_β has to have infinite intersection with each member of a certain collection of less than 2^ω infinite subsets of ω , and $\omega - X_\beta$ also has to have infinite intersection with each member of a certain collection (not necessarily the same) of less than

2^ω infinite subsets of ω . (Since, in this case, we can pick each Y_η to be a singleton, we take appropriate unions of the Y_η 's for each pair i, ξ .) It is a well-known consequence of MA that this is possible (cf. Remark 7.2). \square

7.1. *Remark.* There exists in ZFC a family $\langle X_\xi: \xi < 2^\kappa \rangle$ satisfying only (1) [H].

7.2. *Remark.* Since initial ω -compactness is not finitely productive in ZFC, and since MA frequently makes cardinals κ with $\omega < \kappa < \mathfrak{c}$ look like ω , one may feel that Theorem 1.6 (initial ω_ω -compactness is not finitely productive under $\text{MA} + \mathfrak{c} > \omega_\omega$) is not too surprising, and one may wonder if there can be a singular $\kappa \geq \mathfrak{c}$ such that initial κ -compactness is not finitely productive. This turns out to be the case. We need a generalization of a weak form of MA for higher cardinals.

For a cardinal κ consider the partial order $\mathbb{D}_\kappa = \langle \mathbb{D}_\kappa, \leq \rangle$, defined by

$$\begin{aligned} \mathbb{D}_\kappa &= \{ \langle A, B \rangle: A, B \in [\kappa]^{<\kappa} \text{ are Disjoint} \}, \\ \langle A, B \rangle &\leq \langle C, D \rangle \text{ if } A \supseteq C \text{ and } B \supseteq D. \end{aligned}$$

Consider the statement

Δ_κ : $2^\lambda = \lambda^+$ if $\omega \leq \lambda < \kappa$, and if \mathcal{D} is a collection of dense subsets of \mathbb{D}_κ with $|\mathcal{D}| < 2^\kappa$, then there is a filter on \mathbb{D}_κ which intersects every member of \mathcal{D} .

Clearly MA implies Δ_ω . Baumgartner proved (more than) that it is consistent with ZFC that $\Delta_{\omega_1} + 2^{\omega_1}$ is as large as you like [B, Theorem 4.2]; the same proof shows that if κ is regular, then it is consistent with ZFC that $\Delta_\kappa + 2^\kappa$ is as large as you like.

It is easy to see that our construction works under Δ_κ . If there is a singular λ with $\kappa < \lambda < 2^\kappa$, and 2^κ is regular, then this yields two initially λ -compact normal spaces whose product is not initially λ -compact.

8. PROOF THAT $E_0 \times E_1$ IS NORMAL IF $\kappa = \omega$

Recall that we are assuming MA, so \mathfrak{c} is regular.

Since the points of $\omega \times \omega$ are isolated, it suffices to prove that every two disjoint closed subsets of

$$X = (E_0 \times E_1) - (\omega \times \omega) = (\Sigma_0 \times \Sigma_1) \cup (\omega \times \Sigma_1) \cup (\Sigma_0 \times \omega)$$

have disjoint closures in

$$\overline{X} = (b\omega \times b\omega) - (\omega \times \omega) = ({}^c 2 \times {}^c 2) \cup (\omega \times {}^c 2) \cup ({}^c 2 \times \omega).$$

Recall from §5, in particular from Remark 5.4, that

- (1) Σ_i is ω -bounded ($i < 2$);
- (2) every two disjoint closed subsets of Σ_i have disjoint closures in ${}^c 2$ ($i < 2$); and
- (3) every two disjoint closed subsets of $\Sigma_0 \times \Sigma_1$ have disjoint closures in ${}^c 2 \times {}^c 2$.

Let F and G be any two disjoint closed subsets of X . Since X has no isolated points we cannot simplify the argument by considering the nonisolated points only, as we did when proving that E_0 and E_1 are normal. However, we use a similar idea: we look at the more nonisolated points first.

Claim. $\overline{F} \cap \overline{G} \cap {}^c 2 \times {}^c 2 = \emptyset$.

Assuming that the Claim holds, it is easy to see from (2) and the above formula for \overline{X} that $\overline{F} \cap \overline{G} = \emptyset$.

To prove the Claim, let $\langle a, b \rangle \in {}^c 2 \times {}^c 2$ be arbitrary. Then

$$\langle a, b \rangle \notin (F \cap (\Sigma_0 \times \Sigma_1))^- \cap (G \cap (\Sigma_0 \times \Sigma_1))^- ,$$

by (3). Hence we can find open A, B in $b\omega$ with $a \in A$, $b \in B$ such that $\overline{A} \times \overline{B}$ does not intersect both $F \cap (\Sigma_0 \times \Sigma_1)$ and $G \cap (\Sigma_0 \times \Sigma_1)$. Without loss of generality

$$(4) \quad (\overline{A} \times \overline{B}) \cap F \cap (\Sigma_0 \times \Sigma_1) = \emptyset .$$

We assert that for some $n < \omega$ we have

$$[(\overline{A} - n) \times (\overline{B} - n)] \cap F = \emptyset .$$

Clearly this assertion implies $\langle a, b \rangle \notin \overline{F}$, hence $\langle a, b \rangle \notin \overline{F} \cap \overline{G}$.

Suppose our assertion to be false. Then there is a strictly increasing sequence $\langle k_n \rangle_n$ in ω and a sequence $\langle x_n \rangle_n$ in ${}^c 2$ such that one of the following holds:

- (a) $\langle k_n, x_n \rangle \in F \cap (A \times B)$ and $x_n \in \Sigma_1$, for all $n < \omega$; or
- (b) $\langle x_n, k_n \rangle \in F \cap (A \times B)$ and $x_n \in \Sigma_0$, for all $n < \omega$.

Without loss of generality we assume that (a) holds. Since $E_0 = \omega \cup \Sigma_0$ is countably compact, it follows from (1) that $E_0 \times \Sigma_1$ is countably compact. Hence the $\langle k_n, x_n \rangle$'s have a cluster point $\langle s, t \rangle$ in $E_0 \times \Sigma_1$. Since the k_n 's are distinct, we have $s \in \Sigma_0$. This contradicts (4) since clearly $\langle s, t \rangle \in F$. This completes the proof. \square

8.1. Remark. Unlike before it is not true that $\beta(E_0 \times E_1) = b\omega \times b\omega$, for $\{\langle n, n \rangle : n < \omega\}$ and $\Sigma_0 \times \Sigma_1$ are disjoint closed subsets of $E_0 \times E_1$ which do not have disjoint closures in $b\omega \times b\omega$. (Alternatively, since $E_0 \times E_1$ is not pseudocompact one can apply Glicksberg's Theorem [G, Theorem 1].)

8.2. Remark. The above argument also shows that $E_i \times E_i$ is normal and that $(E_i \times E_i) - (\omega \times \omega)$ is countably compact ($i < 2$). It follows that if $E = E_0 + E_1$, then E is countably compact, and $E \times E$ is normal and countably paracompact but not pseudocompact ($E \times E$ is countably paracompact since its subspace of nonisolated points is countably compact). This gives a consistent counterexample to two questions: Keesling asked if all finite powers of X are pseudocompact if X is countably compact and normal [K], and Parsons asked if $X \times X$ is countably compact if X is countably compact and normal and $X \times X$ is countably paracompact [P]; Parsons gives an honest example of a countably compact X such that $X \times X$ is countably paracompact but not countably compact. I am indebted to Jerry Vaughan for bringing these questions to my attention.

8.3. Remark. X has two "levels": $\omega \times \Sigma_1 \cup \Sigma_0 \times \omega$ and $\Sigma_0 \times \Sigma_1$. I was not interested in attempting the four level proof that $E_0 \times E_1$ is normal for $\kappa > \omega$.

REFERENCES

- [B] J. E. Baumgartner, *Iterated forcing*, Surveys in Set Theory (A. R. D. Mathias, ed.), Cambridge Univ. Press, 1983, pp. 1–59.
- [C] H. H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. **81** (1959), 785–796.
- [vD] E. K. van Douwen, *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (1980), 417–427.

- [FG] N. J. Fine and L. Gillman, *Extension of continuous functions in βN* , Bull. Amer. Math. Soc. **13** (1962), 29–36.
- [Fr] Z. Frolik, *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1963), 87–91.
- [G] I. Glicksberg, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. **90** (1959), 369–382.
- [GFW] S. L. Gulden, W. M. F. Fleischman, and J. H. Weston, *Linearly ordered topological spaces*, Proc. Amer. Math. Soc. **24** (1970), 197–203.
- [H] F. Hausdorff, *Über zwei Sätze von G. Fichtenholz und L. Kantorovich*, Studia Math. **6** (1936), 18–19.
- [K] J. Keesling, *Normality and compactness are equivalent in hyperspaces*, Bull. Amer. Math. Soc. **76** (1970), 618–619.
- [Ko] A. P. Kombarav, *On the normality of Σ_m -products*, Dokl. Akad. Nauk SSSR **211** (1973), 524–527 = Soviet Math. Dokl. **14** (1973), 1050–1054.
- [KoM] A. P. Kombarova and V. I. Malyhin, *On Σ -products*, Dokl. Akad. Nauk SSSR **211** (1973), 774–776 = Soviet Math. Dokl. **14** (1973), 1780–1783.
- [Ku] K. Kunen, *Set theory*, North-Holland, Amsterdam, 1980.
- [N] J. Novák, *On the cartesian product of two compact spaces*, Fund. Math. **40** (1953), 106–112.
- [P] L. Parsons, *An example of a space which is countably compact whose square is countably paracompact but not countably compact*, Proc. Amer. Math. Soc. **65** (1977), 351–354.
- [R] M. E. Rudin, *Lecture notes in set-theoretic topology*, Regional Conf. Ser. in Math., no. 23, Amer. Math. Soc., Providence, R.I., 1973.
- [SS] V. Saks and R. M. Stephenson, *Products of m -compact spaces*, Proc. Amer. Math. Soc. **28** (1971), 279–328.
- [SV] R. M. Stephenson and J. E. Vaughan, *Products of initially m -compact spaces*, Trans. Amer. Math. Soc. **196** (1974), 177–189.
- [T] H. Teresaka, *On cartesian product of compact spaces*, Osaka J. Math. **4** (1952), 11–15.

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